3.1 Quadratic Functions

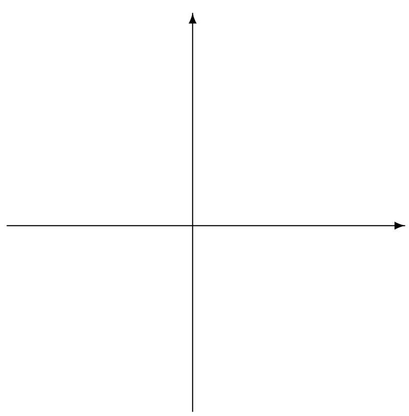
The Graph of a Quadratic Function

The general form of a quadratic function is $y = ax^2 + bx + c$. We have already studied the graph of the parabola $y = x^2$. We have also seen that through translations, such a graph can be shifted up or down, right or left, stretched, compressed, or reflected about the x-axis.

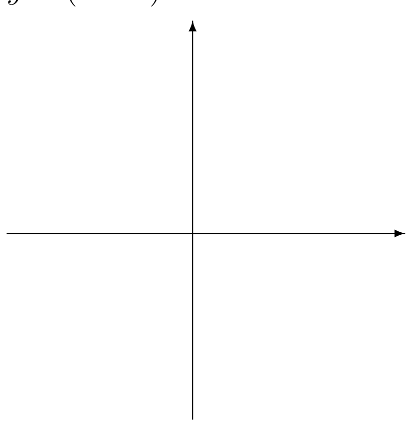
The graph of a quadratic function is called a parabola. The vertex is the point where the parabola achieves its minimum, if it is upward facing, or its maximum, if it is downward facing.

Example Sketch the graph. Label the vertex.

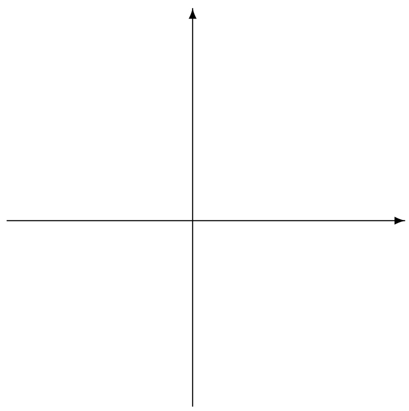
1. $y = x^2$



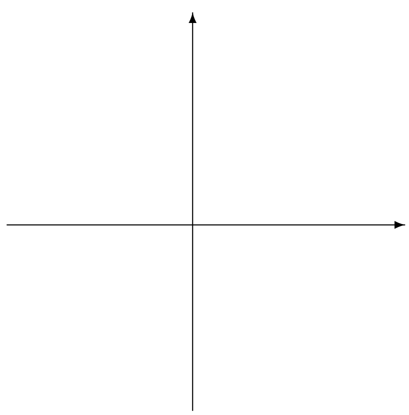
2. $y = (x - 2)^2$



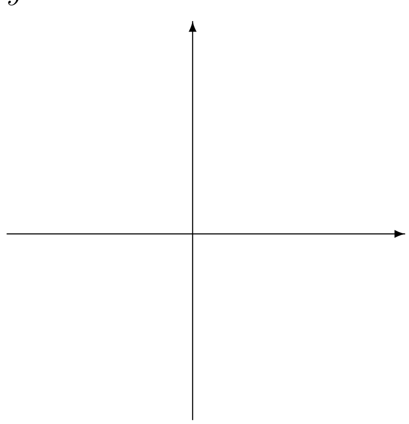
3. $y = x^2 + 3$



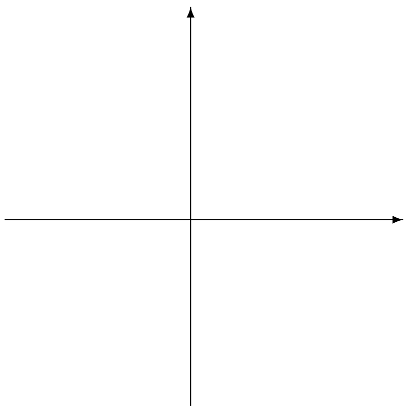
4. $y = 4x^2$



5. $y = -x^2$



6. $y = -(x-2)^2 + 1$



The Standard Form of a Quadratic Equation

The standard form of a quadratic function is

$$y = a(x-h)^2 + k$$

where the vertex is at the point (h, k). If $a > 0$, then the parabola is upward facing, If $a < 0$, then the parabola is downward facing.

Suppose we are given a quadratic function in the general form $y = ax^2 + bx + c$ and we want to write it in the standard form $y = a(x-h)^2 + k$. There are a couple of ways to do this. Let's first use the method of **completing the square.**

When we use the method of completing the square, then we should keep in mind that $(x-h)^2$ can be expanded to give

$$(x-h)^2 = x^2 - 2hx + h^2$$

Example Write the quadratic function $y = x^2 - 8x$ in the form $(x-h)^2 + k$.

Solution We see that the coefficient of x is the number -8. The coefficient of x corresponds to $-2h$ in the expansion shown above. So $-2h = -8$, and $h = 4$, $h^2 = 16$. We have

$$y = (x^2 - 8x + 16) - 16$$

We added $h^2 = 16$ and then subtracted 16. By adding and subtracting 16, we have only changed the form of the function. It is still the same function.

We can find $h^2 = 16$ by dividing the coefficient of x by 2 and then squaring it. In our case $\left(-\frac{8}{2}\right)^2 = 16$.

We now factor.

$$\begin{aligned} y &= (x^2 - 8x + 16) - 16 \\ y &= (x - 4)^2 - 16 \end{aligned}$$

We are done. The vertex of the parabola is $(4, 16)$. □

Example Write the quadratic function in the form $y = a(x - h)^2 + k$ by completing the square.

$$y = x^2 + 6x + 2$$

Solution

$$\begin{aligned} y &= x^2 + 6x + 2 \\ y &= (x^2 + 6x + \underbrace{9}_{\left(\frac{6}{2}\right)^2}) - 9 + 2 \\ y &= (x + 3)^2 - 7 \end{aligned}$$

□

Completing the square can be more difficult if the coefficient of x^2 is not 1 and the coefficient of x is not an even number.

Example Write the function $y = -2x^2 + 5x + 1$ in the form $y = a(x - h)^2 + k$ by completing the square.

Solution First we have to factor out the coefficient of x^2 from the first two terms.

$$y = -2\left(x^2 - \frac{5}{2}x\right) + 1$$

We now look at the expression $x^2 - \frac{5}{2}x$ and complete the square. We divide $\frac{5}{2}$ by 2 and then square it to get $\frac{25}{16}$. We then add and subtract $\frac{25}{16}$

$$y = -2\left(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}\right) + 1$$

We now bring $-\frac{25}{16}$ outside the parenthesis. Note that we need to multiply it by -2.

$$y = -2\left(x^2 - \frac{5}{2}x + \frac{25}{16}\right) - 2 \cdot \left(-\frac{25}{16}\right) + 1$$

$$y = -2\left(x - \frac{5}{4}\right)^2 + \frac{25}{8} + \frac{8}{8}$$

$$y = -2\left(x - \frac{5}{4}\right)^2 + \frac{33}{8}$$

The vertex of the parabola is $\left(\frac{5}{4}, \frac{33}{8}\right)$ and it faces downward. □

Completing the square is especially impractical if the coefficients are not integers. Instead of completing the square, we can use the formula for h. Given the general form of a quadratic $f(x) = ax^2 + bx + c$, then

$$h = -\frac{b}{2a}$$

We can derive this formula by completing the square of the general form $y = ax^2 + bx + c$. We will not show that here. Try to do this as an exercise.

Note that k is the value of the function when x is equal to h. To find k, just put h back into the original function

$$k = f(h)$$

The value for a was already given in the general form $f(x) = ax^2 + bx + c$, and this will be the same number for the standard form $y = a(x - h)^2 + k$.

Let's now do the previous example again, but this time using the formula for h instead of completing the square.

Example Write the function $y = -2x^2+5x+1$ in the form $y = a(x-h)^2+k$ by using the formula $h = -\frac{b}{2a}$.

SOLUTION

$$a = -2, \quad b = 5, \quad c = 1$$

$$h = -\frac{b}{2a} = -\frac{5}{(2)(-2)} = -\frac{5}{4}$$

$$\begin{aligned} k &= f\left(-\frac{5}{4}\right) \\ &= -2\left(-\frac{5}{4}\right)^2 + 5\left(-\frac{5}{4}\right) + 1 \\ &= -2 \cdot \frac{25}{16} - \frac{25}{4} + 1 \\ &= -\frac{25}{8} + \frac{50}{8} + \frac{8}{8} = \frac{33}{8} \end{aligned}$$

We have $a = -2$, $h = -\frac{5}{4}$, and $k = \frac{33}{8}$. We put these values into the standard form $y = a(x-h)^2 + k$.

$$y = -2\left(x + \frac{5}{4}\right)^2 + \frac{33}{8} \qquad \square$$

The Zeros of a Quadratic Function

The x-intercepts of the graph of the quadratic function are the zeros of the function. To find the zeros of a quadratic function $y = ax^2 + bx + c$, we must solve the equation

$$ax^2 + bx + c = 0.$$

If the quadratic factors over the integers, then factoring will be the quickest method in finding the zeros.

Example Find the zeros of the quadratic function $y = x^2 + 5x + 6$.

SOLUTION

$$\begin{aligned} x^2 + 5x + 6 &= 0 \\ (x+2)(x+3) &= 0 \end{aligned}$$

$x = -2$ or $x = -3$. □

If the quadratic does not factor over the integers, then we can use the quadratic formula.

The Quadratic Formula The quadratic equation

$$ax^2 + bx + c = 0$$

has real solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided $b^2 - 4ac > 0$.

Example Use the quadratic formula to find the zeros of the quadratic function $y = 2x^2 + 5x + 1$.

SOLUTION

$$a = 2, \quad b = 5, \quad c = 1$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-5 \pm \sqrt{(5)^2 - 4(2)(1)}}{2(2)} \\ &= \frac{-5 \pm \sqrt{25 - 8}}{4} \\ &= \frac{-5 \pm \sqrt{17}}{4} \qquad \square \end{aligned}$$

If $b^2 - 4ac$ is negative, then there is a negative number under the square root. The square root of a negative number is not a real number. Therefore, there will be no real solutions, and the parabola will not have any x-intercepts.

We define the imaginary unit as $i = \sqrt{-1}$. A complex number is of the form $a+bi$ where a and b are real numbers. We will not be studying complex numbers in this section.

Example Let $f(x) = -2x^2 + 12x + 14$.

1. Write the function in the standard form $y = a(x-h)^2 + k$.

   SOLUTION $a = -2,\ b = 12,\ c = 14$

   $$\begin{array}{lll} h & = 7 - \frac{b}{2a} = -\frac{12}{2(-2)} = 3 \\ k & = & f(h) = f(3) = -2(3)^2 + 12(3) + 14 = 32 \end{array}$$

   We put $a = -2,\ h = 3,\ k = 32$ into the standard form $y = a(x-h)^2+k$.

   $$y = -2(x-3)^2 + 32$$

2. State the vertex.

   SOLUTION $(3, 32)$.

3. Find the y-intercept.

   SOLUTION We set $x = 0$ and solve for y. If we start with $f(x) = -2x^2 + 12x + 14$, and put in $x = 0$, then all the x terms disappear and we get $y = 14$. The y-intercept is $(0, 14)$.

4. Find the x-intercepts.

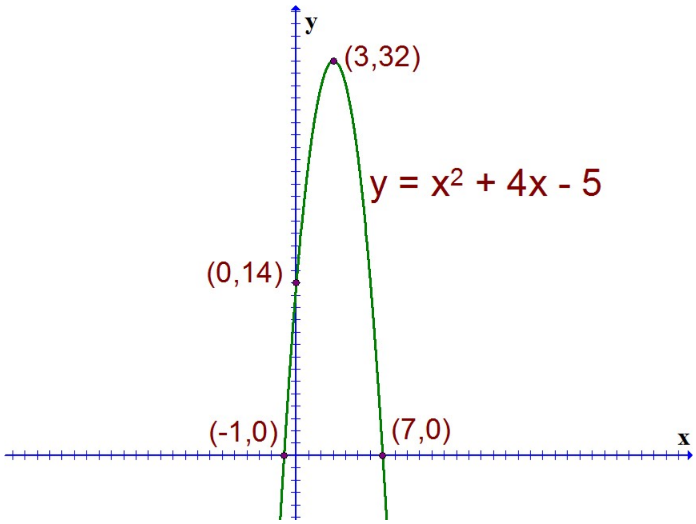
   SOLUTION Solve

   $$\begin{aligned} -2x^2 + 12x + 14 &= 0 \\ -2(x^2 - 6x - 7) &= 0 \\ -2(x-7)(x+1) &= 0 \\ x = 7,\ x = -2 & \end{aligned}$$

   The x-intercepts are $(7, 0)$ and $(-2, 0)$.

5. Sketch the graph.

   SOLUTION



Quadratic Inequalities

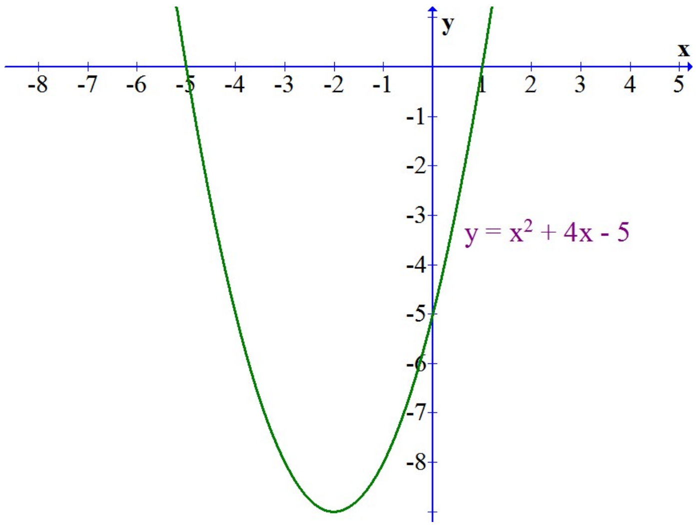
Example Solve each inequality and write the solution set in interval notation.

$$x^2 + 4x - 5 > 0$$

SOLUTION We draw a rough sketch of the graph $y = x^2 + 4x - 5$. We find the zeros of the quadratic.

$$\begin{aligned} x^2 + 4x - 5 &= 0 \\ (x+5)(x-1) &= 0 \\ x = -5,\ x = 1 & \end{aligned}$$

Note that the graph faces upward because the coefficient of x^2 is positive. It is not necessary to find the vertex to solve this problem.



We look at the graph and determine for which values of x the function lies above the x-axis. The answer is $(-\infty, -5) \cup (1, \infty)$. □

The Points of Intersection of a Quadratic and Linear Function

Example Find the point of intersection of the parabola and the straight line given by the following equations. Sketch the graph of the line and the parabola on the same plane.

$$y = x^2 + 6x - 10, \quad y = 2x + 2$$

SOLUTION

We set the two equations equal to each other and then solve for x.

$$\begin{aligned} x^2 + 6x - 10 &= 2x + 2 \\ x^2 + 4x - 12 &= 0 \\ (x+6)(x-2) &= 0 \\ x = -6, \quad & \quad x = 2 \end{aligned}$$

To find the y-coordinates, we substitute $x = -6$ and $x = -2$ into either function. When $x = -6$, $y = 2(-6)+2 = -10$. When $x = 2$, $y = 2(2)+2 = 6$. The points of intersection are $(-6, -10)$ and $(2, 6)$

To sketch the graph of the parabola, we can complete the square and write the function in standard form, $y = a(x-h)^2 + k$.

$$y = x^2 + 6x - 10$$
$$y = (x^2 + 6x + 9) - 9 - 10$$
$$y = (x+3)^2 - 19$$

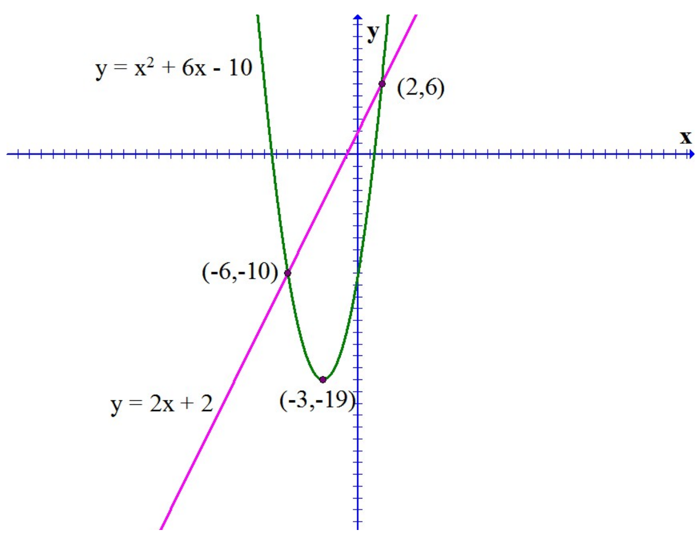
The graph is an upward facing parabola with vertex $(-3, -19)$.

The quadratic does not factor, therefore, the zeros will be irrational numbers. $a = 1,\ b = 6,\ c = -10$

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{(6)^2 - 4(1)(-10)}}{2(1)} \\ &= \frac{-6 \pm \sqrt{76}}{2} = \frac{-6 \pm \sqrt{4 * 19}}{2} \\ &= \frac{-6 \pm 2\sqrt{19}}{2} = \frac{2(-3 \pm \sqrt{19})}{2} \\ x &= -3 \pm \sqrt{19} \end{aligned}$$

$$x = -3 + \sqrt{19} \approx 1.3589, \quad x = -3 - \sqrt{19} \approx -7.3589$$

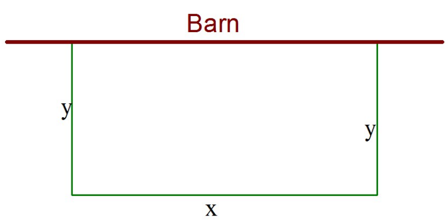
The line $y = 2x + 2$ has slope $m = 2$ and y-intercept $(0, 2)$.



Applications of Maximum and Minimum

Example A farmer has 140 feet of fencing to construct a rectangular enclosure. She will use the side of the barn as one side of the enclosure, so she only needs the fencing for the remaining three sides. What dimensions will maximize the area of the enclosure?

SOLUTION



- The total amount of fencing is given by the constraint

$$x + 2y = 140$$

- The area of the rectangle is given by

$$A = xy$$

- We eliminate the variable x.

$$\begin{aligned} x + 2y &= 140 \\ x &= -2y + 140 \\ A = xy = (-2y + 140)y \\ A = -2y^2 + 140y \end{aligned}$$

- To find the value of y where the area, A, is at its maximum, we find the first coordinate, h, of the vertex of the parabola $A = -2y^2 + 140y$.

$$h = -\frac{b}{2a} = -\frac{140}{2(-2)} = 35 \text{ ft}$$

So $y = 35$ feet will give a maximum area.

- We solve for x by putting in $y = 35.5$ into the equation $x + 2y = 140$. We have

$$\begin{aligned} x + 2y &= 140 \\ x + 2(35.5) &= 140 \\ x = 70 \text{ feet} \end{aligned}$$

- The dimensions that will give the maximum area are 70 feet by 35.5 feet. $\square$