3.2 Zeros of Polynomials

A function f defined for all numbers is called a **polynomial function** if there exists numbers $a_0, a_1, a_2, \ldots, a_n$ such that for all numbers s we have

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

The numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are uniquely determined. They will be called the **coefficients** of f, and we call a_n the **leading coefficient** if $a_n \neq 0$. We call a_0 the **constant term**. If $a_n \neq 0$, then we say that the polynomial has **degree** n. We denote the degree of a polynomial f as $\deg f$.

Example. State the degree of the polynomial.

1. $f(x) = -5x^3 + 2x + 1$

   Solution $\deg f = 3$

2. $f(x) = (x-2)(x+1)(x+3)$

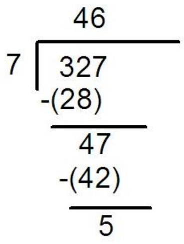
   Solution $\deg f = 3$

3. $f(x) = (x-1)^2(x+3)^4$

   Solution $\deg f = 6$

Long Division and the Euclidean Algorithm

Example. Divide 327 by 7, with a possible remainder. The steps in long division are divide, multiply, subtract, and bring down.



$$\frac{327}{7} = 46 + \frac{5}{7}$$

$$327 = 7 \cdot 46 + 5$$

Euclidean Algorithm for Integers. Let n, d be positive integers. Then there exists an integer r such that $0 \leq r < d$, and an integer $q \geq 0$ such that

$$n = qd + r$$

Euclidean Algorithm for Polynomials. Let f, g be non-zero polynomials. Then there exists polynomials q, r such that $\deg r < \deg g$ and such that

$$f(x) = q(x)g(x) + r(x)$$

Example. Let $f(x) = 4x^3 - 3x^2 + x + 2$ and $g(x) = x^2 + 1$. Find $q(x)$ and $r(x)$ guaranteed by the Euclidean algorithm.

SOLUTION

$$\begin{array}{r|rrrr}
 & & & 4x & -3 \\ \hline
x^2+1 & 4x^3 & -3x^2 & +x & +2 \\
 & -(4x^3 & & +4x) & \\ \cline{2-3}
 & & -3x^2 & -3x & +2 \\
 & & -(-3x^2 & & -3) \\ \cline{3-5}
 & & & -3x & +5
\end{array}$$

$q(x) = 4x - 3$, $r(x) = -3x + 5$.

$$f(x) = (4x - 3)(x^2 + 1) + (-3x + 5) \quad \square$$

We do not prove the Euclidean algorithm. The proof would consist of carrying out the procedure of the example with general coefficients.

Zeros of Polynomials

The roots of a quadratic can always be found using the quadratic formula. If a polynomial has a higher degree, then it is much more difficult to determine the roots, except in very special cases. For polynomials of degrees 3 and 4, the roots can be found by using formulas involving radicals, but **it is a classical result that such formulas cannot be given in general for polynomials of degree at least 5.**

Theorem. If $f(x)$ has a root c, that is, $f(c) = 0$, then we can write

$$f(x) = (x - c)q(x)$$

for some polynomial q.

PROOF. By the Euclidean algorithm, we have

$$f(x) = q(x)(x - c) + r(x),$$

where $\deg r < \deg(x-c)$, that is, $\deg r < 1$. Therefore, r must be a constant, say equal to some number a. Then

$$f(x) = q(x)(x - c) + a$$

Now we put in $x = c$, a root of f, and get

$$f(c) = q(c)(c - c) + a$$

$$0 = 0 + a$$

We see that $a = 0$ so that $f(x) = (x - c)q(x)$. $\square$

Example. Show that $c = 2$ is a root of the polynomial $f(x) = 3x^3 - 11x^2 + 11x - 2$, then factor f completely.

SOLUTION We have $f(2) = 3(2)^3 - 11(2)^2 + 11(2) - 2 = 24 - 44 + 22 - 2 = 0$. Therefore, $(x - 2)$ is a factor of f and using long division we can find a q such that $f(x) = (x - 2)q(x)$. We get $q(x) = 3x^2 - 5x + 1$, so

$$f(x) = (x - 2)(3x^2 - 5x + 1)$$

The roots of q are given by the quadratic formula,

$$c_1 = \frac{5+\sqrt{25-12}}{6} = \frac{5+\sqrt{13}}{6}$$

$$c_2 = \frac{5-\sqrt{25-12}}{6} = \frac{5-\sqrt{13}}{6}$$

Therefore $f(x) = a(x-2)(x-c_1)(x-c_2)$ where a is a constant. When f is expanded, we see that a is the lead coefficient, so $a = 3$. We then have the complete factorization of f.

$$f(x) = 3(x-2)\left(x - \frac{5+\sqrt{13}}{6}\right)\left(x - \frac{5-\sqrt{13}}{6}\right) \qquad \square$$

REMARK. If f is a quadratic in general form

$$f(x) = ax^2 + bx + c,$$

and $a \neq 0$ and $b^2 - 4ac > 0$, then there are two distinct roots c_1 and c_2 of f, and therefore we have the factorization

$$f(x) = a(x-c_1)(x-c_2).$$

Example. Find the complete factorization of the polynomial

$$f(x) = x^n - 1.$$

SOLUTION We see that 1 is a root, because $f(1) = 1^n - 1 = 1 - 1 = 0$. Hence we know that f must have $(x-1)$ as a factor, i.e., we have

$$f(x) = (x-1)q(x)$$

for some $q(x)$. We can find q using long division. We have $q(x) = x^{n-1} + x^{n-2} + \ldots + x + 1$. That is

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1)$$

In particular, when $n = 3$, we have

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

As an exercise, try to find a factorization for $f(x) = x^n - a^n$.

The Rational Root Theorem. If f is a polynomial with integer coefficients $a_n, a_{n-1}, \ldots, a_1, a_0$, that is

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_0$$

then if f has a rational root of the form $\frac{p}{q}$ where p and q are integers that have no common factors, then p is a factor the constant coefficient a_0, q is a factor of the lead coefficient a_n, allowing positive or negative signs for p and q.

No proof is given here, not because the proof is especially difficult, but because I am trying to save space on the page.

We can try to factor a higher degree polynomial by trying to guess the roots of the polynomial. If the polynomial does have a rational root, which is rarely the case except for problems from textbooks, we can use the Rational Root Theorem to narrow down the choices for our guess.