3.2 Zeros of Polynomials

3.5 Graphs of Polynomials

1. What is the degree of the following polynomials?

   (a) $3x^2 - 4x + 5$ 2

   (b) $-5x^5 + 5$ 5

   (c) $-38x^4 + x^3 - x - 1$ 4

   (d) $(3x^2 - 4x + 5)(-5x^4 + x)$ 6

   (e) $(-5x^5 + x)(-7x + 3)$ 6

   (f) $(-4x^2 + 5x - 4)(3x^3 + x - 1)$ 5

   (g) $(6x^7 - x^3 + 5)(7x^4 - 3x^2 + x - 1)$ 11

2. Find the polynomial $q(x)$ and $r(x)$ of the Euclidean algorithm when $f(x) = 4x^3 - x + 2$, and:

   (a) $g(x) = x - 2$

   Solution

   $$\begin{array}{r|l} & 4x^2 + 8x + 15 \\ \hline x - 2 & 4x^3 + 0x^2 - x + 2 \\ & -(4x^3 - 8x^2) \\ \hline & 8x^2 - x + 2 \\ & -(8x^2 - 16x) \\ \hline & 15x + 2 \\ & -(15x - 30) \\ \hline & 32 \end{array}$$

   $$q(x) = 4x^2 + 8x + 15, \quad r(x) = 32$$

   $$4x^3 - x - 2 = (x - 2)(4x^2 + 8x + 15) + 32$$

   (b) $g(x) = x^2 - 1$

   $$q(x) = 4x, \quad r(x) = 3x + 2$$

   $$4x^3 - x + 2 = 4x(x^2 - 1) + (3x + 2)$$

(c) $g(x) = x^2 + 1$

$$q(x) = 4x, \quad r(x) = -5x + 2$$

$$4x^3 - x + 2 = 4x(x^2 + 1) + (-5x + 2)$$

(d) $g(x) = x^2 - x$

$$q(x) = 4x + 4, \quad r(x) = 3x + 2$$

$$4x^3 - x + 2 = (4x + 4)(x^2 - x) + (3x + 2)$$

(e) $g(x) = x^2 - x + 1$

$$q(x) = 4x + 4, \quad r(x) = -x - 2$$

$$4x^3 - x + 2 = (4x + 4)(x^2 - x + 1) + (-x - 2)$$

(f) $g(x) = x^2 + x - 1$

$$q(x) = 4x - 4, \quad r(x) = 7x - 2$$

$$4x^3 - x + 2 = (4x - 4)(x^2 + x - 1) + (7x - 2)$$

(g) $g(x) = x^3 + 2$

$$q(x) = 4, \quad r(x) = -x - 6$$

$$4x^3 - x + 2 = (4)(x^3 + 2) + (-x - 6)$$

(h) $g(x) = x^3 - x + 1$

$$q(x) = 4, \quad r(x) = 3x - 2$$

$$4x^3 - x + 2 = (4)(x^3 - x + 1) + (3x - 2)$$

3. Show that the given number c is a root of the given polynomial f, then factor f completely over the real numbers, if it is possible.

(a) $f(x) = 2x^3 - 3x^2 - 8x - 3, \quad c = -1$

SOLUTION Because $c = -1$ is a root of f, there exists a polynomial $q(x)$ such that

$$f(x) = (x+1)q(x)$$

To find q, we use long division.

```
              2 x² - 5 x - 3
      ______________________
x +1 | 2 x³ - 3 x² - 8 x - 3
      -(2x³ + 2x²)
      ____________
             -5x² - 8x - 3
           -(-5x² - 5x)
           _____________
                  -3x-3
                -(-3x-3)
                 _______
                      0
```

So $q(x) = 2x^2 - 5x - 3$ and

$$f(x) = (x+1)(2x^2 - 5x - 3)$$

We now factor q.

$$2x^2 - 5x - 3 = 0$$

We can factor directly or we can find the roots using the quadratic formula.

$$x = \frac{5 \pm \sqrt{(5)^2 - 4(-3)(2)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{49}}{4}$$

$$x = \frac{5 \pm 7}{4}$$

$$c_1 = -\frac{1}{2}, \quad c_2 = 3$$

$$f(x) = 2(x+1)\left(x + \frac{1}{2}\right)(x-3)$$

$$f(x) = (x+1)(2x+1)(x-3)$$

(b) $f(x) = x^3 - 3x^2 - 18x + 40$, $c = 2$

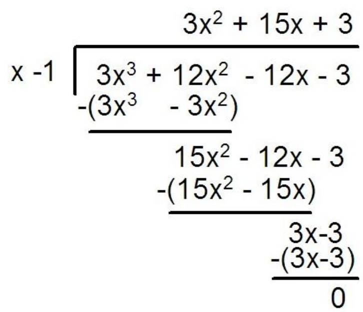
SOLUTION $f(x) = (x-2)(x+4)(x-5)$

(c) $f(x) = 6x^3 + 3x^2 - 39x + 18$, $c = -3$

SOLUTION $f(x) = 3(x-2)(x+3)(2x-1)$

(d) $f(x) = 3x^3 + 12x^2 - 12x - 3$, $c = 1$

SOLUTION



$$f(x) = (x-1)(3x^2 + 15x + 3)$$

Find the roots of $q(x) = 3x^2 + 15x + 3$ using the quadratic formula.

$$x = \frac{-15 \pm \sqrt{(15)^2 - 4(3)(3)}}{2(3)}$$

$$x = \frac{-15 \pm \sqrt{189}}{2(3)}$$

$$x = \frac{-15 \pm 3\sqrt{21}}{2(3)}$$

$$x = \frac{3(-5 \pm \sqrt{21})}{2(3)}$$

$$x = \frac{-5 \pm \sqrt{21}}{2}$$

$$c_1 = \frac{-5+\sqrt{21}}{2}, \quad c_2 = \frac{-5-\sqrt{21}}{2}$$
$$f(x) = 3(x-1)(x-c_1)(x-c_2)$$

$$f(x) = 3(x-1)\left(x - \frac{-5+\sqrt{21}}{2}\right)\left(x - \frac{-5-\sqrt{21}}{2}\right)$$

$$f(x) = \frac{3}{4}(x-1)\left(2x+5-\sqrt{21}\right)\left(2x+5+\sqrt{21}\right)$$

(e) $f(x) = 4x^3 - 4x^2 - 44x - 40$, $c = -2$

Solution $f(x) = (2x - \sqrt{29} - 3)(x+2)(2x + \sqrt{29} - 3)$

4. Factor the fourth degree polynomial by writing it as a quadratic. Then find all real and imaginary solutions.

(a) $x^4 - 5x^2 + 4 = 0$

Solution

$$\begin{aligned} x^4 - 5x^2 + 4 &= 0 \\ (x^2)^2 - 5(x^2) + 4 &= 0 \\ u^2 - 5u + 4 &= 0 \\ (u-4)(u-1) &= 0 \\ u = 4 \qquad & u = 1 \\ x^2 = 4 \qquad & x^2 = 1 \\ x = \pm 2 \qquad & x = \pm 1 \end{aligned}$$

(b) $x^4 - 13x^2 + 36 = 0$

SOLUTION

$$\begin{aligned} x^4 - 13x^2 + 36 &= 0 \\ (x^2)^2 - 13(x^2) + 36 &= 0 \\ u^2 - 13u + 36 &= 0 \\ (u-4)(u-9) &= 0 \\ u = 4 &\qquad u = 9 \\ x^2 = 4 &\qquad x^2 = 9 \\ x = \pm 2 &\qquad x = \pm 3 \end{aligned}$$

(c) $x^4 - 7x^2 + 10 = 0$
SOLUTION

$$\begin{aligned} x^4 - 7x^2 + 10 &= 0 \\ (x^2)^2 - 7(x^2) + 10 &= 0 \\ u^2 - 7u + 10 &= 0 \\ (u-2)(u-5) &= 0 \\ u = 2 &\qquad u = 5 \\ x^2 = 2 &\qquad x^2 = 5 \\ x = \pm\sqrt{2} &\qquad x = \pm\sqrt{5} \end{aligned}$$

(d) $x^4 - 1 = 0$
SOLUTION

$$\begin{aligned} x^4 - 1 &= 0 \\ (x^2)^2 - 1 &= 0 \\ u^2 - 1 &= 0 \\ (u-1)(u+1) &= 0 \\ u = 1 &\qquad u = -1 \\ x^2 = 1 &\qquad x^2 = -1 \\ x = \pm 1 &\qquad x = \pm\sqrt{-1} \\ x = \pm 1 &\qquad x = \pm i \end{aligned}$$

(e) $x^4 - 3x^2 - 4 = 0$

SOLUTION

$$\begin{aligned} x^4 - 3x^2 - 4 &= 0 \\ (x^2)^2 - 3(x^2) - 4 &= 0 \\ u^2 - 3u - 4 &= 0 \\ (u-4)(u+1) &= 0 \\ u = 4 \quad & \quad u = -1 \\ x^2 = 4 \quad & \quad x^2 = -1 \\ x = \pm\sqrt{4} \quad & \quad x = \pm\sqrt{-1} \\ x = \pm 2 \quad & \quad x = \pm i \end{aligned}$$

5. For the given polynomial functions,

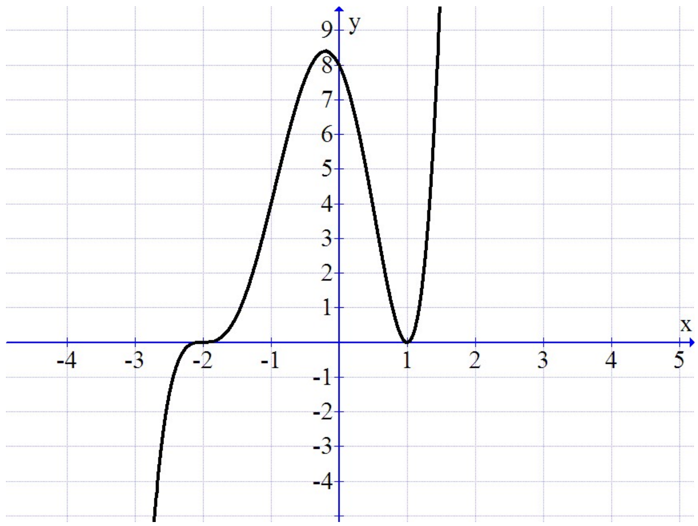
   - Find the zeros and their multiplicity.
   - Find the lead term.
   - Sketch the graph.

   (a) $y = (x-1)^2(x+2)^3$

   SOLUTION
   Zeros:

   - $x = 1$ multiplicity 2, touches
   - $x = -2$ multiplicity 3, passes

   Lead term: $y = x^2 \cdot x^3 = x^5$
   Graph:



(b) $y = (x+1)^2 x^2 (x-2)$

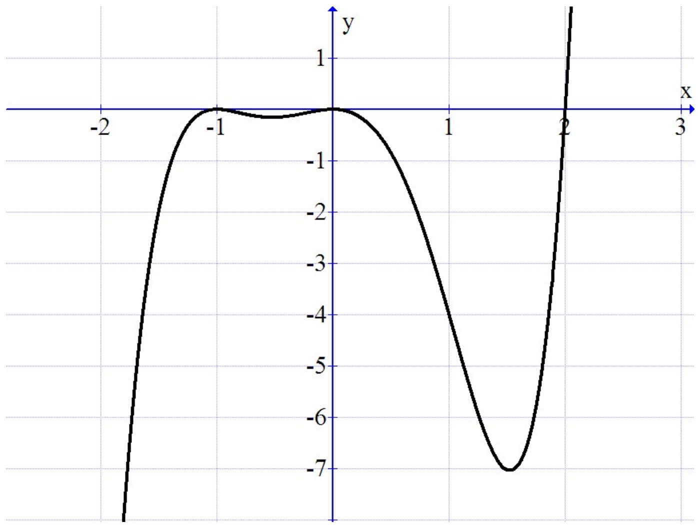
SOLUTION

Zeros:

- $x = -1$ multiplicity 2, touches
- $x = 0$ multiplicity 2, touches
- $x = 2$ multiplicity 1, passes

Lead term: $y = x^2 \cdot x^2 \cdot x = x^5$

Graph:



(c) $y = -(x-1)^2(x+1)^2$

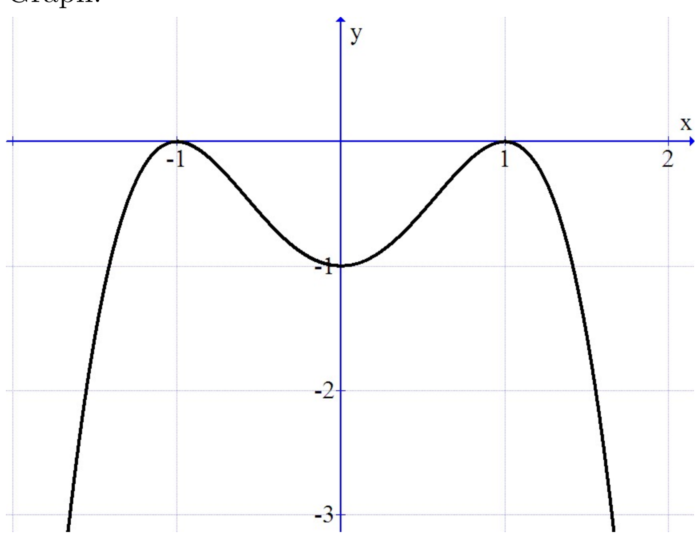
Solution

Zeros:

- $x = 1$ multiplicity 2, touches
- $x = -1$ multiplicity 2, touches

Lead term: $y = x^2 \cdot x^2 = -x^4$

Graph:



(d) $y = (x-2)^3(x+1)^2$

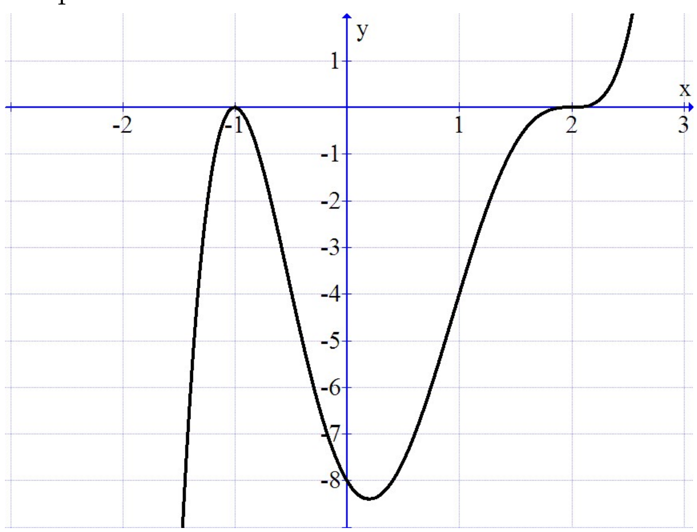
SOLUTION

Zeros:

- $x = 2$ multiplicity 3, passes
- $x = -1$ multiplicity 2, touches

Lead term: $y = x^3 \cdot x^2 = x^5$

Graph:



(e) $y = x^2(x-3)(x+2)$

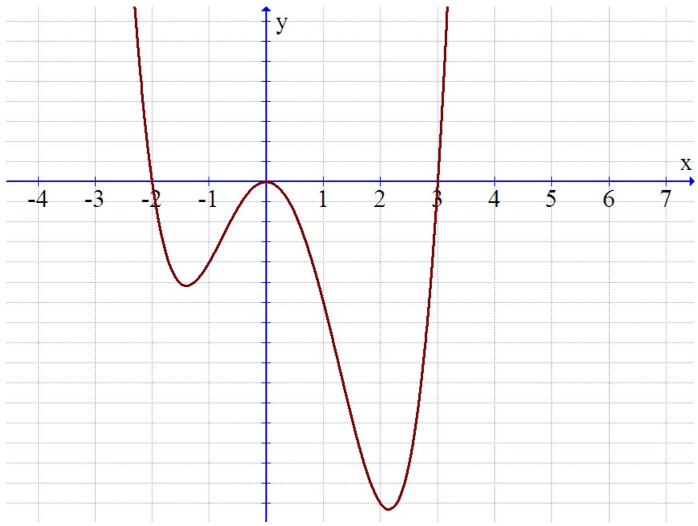
SOLUTION

Zeros:

- $x = 0$ multiplicity 2, touches
- $x = 3$ multiplicity 1, passes
- $x = -2$ multiplicity 1, passes

Lead term: $y = x^2 \cdot x \cdot x = x^4$

Graph:


y
x
-4
-3
-2
-1
1
2
3
4
5
6
7