Notes Math 130, College Algebra

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June 22, 2008

Contents

1 Equations, Inequalities, and Mathematical Modeling

1.2 Solving linear equations in one variable

Homework §1.2 #43-59. See examples 1-4.

A linear equation in one variable is an equation that can be written in the form

$$ax + b = 0$$

Our goal in this section will be to solve linear equations in one variable.

Example 1 Solve $3x - 6 = 0$

SOLUTION

$$\begin{aligned} 3x - 6 &= 0 \\ 3x &= 6 \\ x &= 2 \end{aligned}$$

Example 2 Solve $5x + 4 = 3x - 8$

SOLUTION

$$\begin{aligned} 5x + 4 &= 3x - 8 \\ (5x - 3x) &= (-8 - 4) \\ 2x &= -12 \\ x &= -6 \end{aligned}$$

Example 3 Solve

$$6(x - 1) + 4 = 3(7x + 1)$$

SOLUTION

Use the distributive property to expand the expressions in the parenthesis.

$$\begin{aligned} 6x - 6 + 4 &= 21x + 3 \\ 6x - 2 &= 21x + 3 \\ (6x - 21x) &= (3 + 2) \\ -15x &= 5 \\ x &= -1/3 \end{aligned}$$

Equations that Lead to Linear Equations

Here we will solve equations involving rational expressions (fractions of polynomials) by rewriting them as linear equations.

Example 4 Solve

$$\frac{x}{3} + \frac{3x}{4} = 2$$

SOLUTION Multiply through by a common denominator.

$$\begin{aligned} 12\left(\frac{x}{3} + \frac{3x}{4}\right) &= 12(2) \\ 12\left(\frac{x}{3}\right) + 12\left(\frac{3x}{4}\right) &= 24 \\ 4x + 9x &= 24 \\ 13x &= 24 \\ x &= 24/13 \end{aligned}$$

Example 5 The following equation has no solution.

$$x = x + 1$$

If we subtract x from both sides, we get $0 = 1$, which is a contradiction. Therefore, we conclude that the equation has no solution.

Example 6 Extraneous solutions:

$$\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$$

SOLUTION First, we factor all polynomials, if possible.

$$\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{(x-2)(x+2)}$$

We then multiply through by the common denominator $(x-2)(x+2)$.

$$\begin{aligned}
(x-2)(x+2)\left(\frac{1}{x-2}\right) &= (x-2)(x+2)\left(\frac{3}{x+2} - \frac{6x}{(x-2)(x+2)}\right) \\
(x+2) &= (x-2)(x+2)\left(\frac{3}{x+2}\right) - (x-2)(x+2)\left(\frac{6x}{(x-2)(x+2)}\right) \\
(x+2) &= 3(x-2) - 6x \\
x+2 &= 3x-6-6x \\
x+2 &= -3x-6 \\
x+3x &= -6-2 \\
4x &= -8 \\
x &= -8/4 \\
x &= -2
\end{aligned}$$

This equation has no solutions. If we put $x = 2$ or $x = -2$ into the original equations, we see that the result is fractions with 0 in the denominator. A fraction with denominator 0 is undefined.

1.4 Solving quadratic equations

Homework \$1.4 #7-33, 67-79. See examples 1, 2, 6.

A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0$$

Our goal for this section will be to study several different methods for solving quadratic equations.

Solving by Factoring

Example 1 Solve the quadratic equation by factoring.

$$x^2 - 5x + 6 = 0$$

SOLUTION

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \\ x - 2 = 0 &\qquad x - 3 = 0 \\ x = 2 &\qquad x = 3 \end{aligned}$$

Example 2 Solve the quadratic equation by factoring.

$$2x^2 + 9x + 7 = 3$$

SOLUTION $x = -1/2$, $x = -4$.

Example 3 Solve by factoring.

$$6x^2 - 3x = 0$$

SOLUTION $x = 0$, $x = 1/2$.

Extracting square roots

Example 4 Solve the following quadratic equations by extracting square roots.

1. $4x^2 = 12$

   SOLUTION $x = \pm\sqrt{3}$

2. $(x-3)^2 = 7$

   SOLUTION $x = 3 \pm \sqrt{7}$

Completing the square

I recommend skipping this.
Example 5 Solve by completing the square.

$$x^2 + 2x - 6 = 0$$

SOLUTION $x = -1 \pm \sqrt{7}$

The Quadratic Formula

The solution to the quadratic equation

$$ax^2 + bx + c = 0$$

is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 6 Solve using the quadratic equation.

1. $x^2 + 3x = 9$

   SOLUTION $x = \dfrac{-3 \pm 3\sqrt{5}}{2}$

2. $2x^2 - x - 1 = 0$

1.7 Interval notation, solving linear and absolute value inequalities

- Homework §1.7 #1-15, 19-55. See examples 1-4.
- Interval notation. An interval can be described by an inequality, a sketch on the number line, and by using interval notation.
- Write an inequality that describes the interval. Sketch the interval on a number line.
  1. $(-3, 5]$
  2. $(-3, \infty)$
  3. $[0, 2]$
  4. $(-\infty, \infty)$
- Solving linear inequalities in one variable.
  1. $5x - 7 > 3x + 9$
  2. $1 - \frac{3x}{2} \geq x - 4$
- Double inequalities.

$$-3 \leq 6x - 1 < 3$$

- Absolute value inequalities. First solve absolute value equations.
  1. $|x - 5| = 2$
  2. $|x - 5| < 2$
  3. $|x + 3| \geq 7$

1.8 Solving quadratic, higher order, and rational inequalities.

- Homework §1.8 #1-31, 37-47. See examples 1-4.
- Solving quadratic inequalities.

1. $x^2 - 1 < 0$
2. $x^2 - x - 6 < 0$
3. $x^2 + 1 > 0$
4. $-x^2 - 1 < 0$

- Solving a general polynomial inequality.

$$2x^3 - 3x^2 - 32x > -48$$

- Solving rational inequalities.

$$\frac{2x-7}{x-5} \leq 3$$

2 Functions and Their Graphs

2.1 Analyzing linear equations and graphing linear equations.

- Homework §2.1 #1-77. See examples 1-4.
- The slope-intercept form of a line.

$$y = mx + b$$

- Sketch the line $y = 2x$ by plotting the points for $x = 0$ and $x = 1$. We then can try to understand what m represents. Do more examples.
- Sketch the lines $y = 2x$, $y = 2x + 1$, etc. and try to understand what b means.
- Sketch $y = \frac{1}{2}x + 1$
- Finding the slope of a line.

$$m = \frac{\Delta y}{\Delta x}$$

Do some examples.

- Point-slope form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

- **Example 1** Find the slope-intercept form of the equation of the line that has slope of 3 and passes through the point $(1, -2)$.

- **Example 2** Find the equation of the line through $(1, 2)$ and $(2, 6)$.

- Parallel lines have equal slopes.

  Example 3 Find the equation of the line through $(2, -1)$ and parallel to the line $2x - 3y = 5$.

- Two non-vertical lines are perpendicular if and only if their slopes are negative reciprocals of each other. Draw two lines through the origin, one with point (a, b) and the other with point $(b, -a)$. They look perpendicular. Working with triangles formed by connecting the two points, we see they are perpendicular.

  Example 4 Find the equation of the line through $(2, -1)$ and perpendicular to the line $2x - 3y = 5$.

2.2 Function notation, domain and range of functions, piecewise defined functions.

- Homework §2.2 #13-69. See examples 2-5.

- A function is rule that assigns to each element in the domain a unique element in the range.

- We can have discrete functions defined by listing pairs of elements.

$$\{(1, 2), (1, 3), (2, 5), (3, 7)\}$$

- A relation is a rule that pairs elements in the domain with elements in the range. A relation is not necessarily a function.

- A function can be represented algebraically.

$$f(x) = 2x + 1$$

- A function can be represented graphically.
- Function notation:

  Example 1 Evaluate the function $f(x) = 2x + 1$ for different values of x.

- Piece-wise defined functions.

$$f(x) = \begin{cases} x+1 & \text{for} \quad x > 0 \\ -1 & \text{for} \quad x \leq 0 \end{cases}$$

- Domain of a function: A function often has an implied domain, e.g. $y = \frac{1}{x+1}$ and $y = \sqrt{x}$.

  Find the domain of these functions.

2.3 Vertical line test, domain, range, and zeros of functions, increasing, decreasing, and constant intervals of a function.

- Homework §2.3 #1-23, 31-37. See examples 1-4.
- Studying the graph of a function f to determine the domain, the range, and values $f(a)$ for different a.
- The vertical line test for functions: A graph represents a function if and only if no vertical line intercepts the graph at more than one point.

  Do some examples.

- Zeros of a function. The zeros of a function f are the x-values such that $f(x) = 0$.
- **Example 1** Find the zeros of the following functions.

  1. $f(x) = 3x^2 - x - 10$

2. $g(x) = \sqrt{10 - x^2}$
3. $h(t) = \frac{2t-3}{t+5}$

- Stating the intervals for which a function f is increasing, decreasing, or constant. Do examples.
- Definition of a minimum value of a relative minimum and relative maximum of a function: Draw a picture. Maybe skip this part because there is no homework assigned about this topic. Come back to it with the word problems.

2.4 A library of functions: linear, absolute value, square root, quadratic, cubic, reciprocal, piecewise-defined.

- Homework §2.4 #1-5, 43-49, 53-59 (all).
- Cubic function. $y = x^3$
- Square root function $y = \sqrt{x}$
- Reciprocal function $y = 1/x$
- Absolute value function
- Parabola
- Constant function
- Greatest integer function (optional)
- The graph of a piece-wise defined function.

2.5 Transformations of functions: vertical, horizontal, reflections, stretches.

A new function h is constructed by a translation to a function f.

- Homework §2.5 #1, 2, 9-17, 19-33, 37, 39, 43-53. See examples 1-5.

- Vertical Shift upward: $h(x) = f(x) + c$
- Vertical shift downward: $h(x) = f(x) - c$
- Horizontal shift to the right: $h(x) = f(x - c)$
- Horizontal shift left: $h(x) = f(x + c)$
- Reflection about the x-axis: $h(x) = -f(x)$
- Reflection about the y-axis: $h(x) = f(-x)$

The following are *nonrigid transformations.*

- Vertical stretch: $h(x) = cf(x), \quad 1 < c$
- Vertical compression: $h(x) = cf(x), \quad 0 < c < 1$
- Horizontal stretch: $h(x) = f(ax), \quad 1 < a$
- Horizontal compression: $h(x) = f(ax), \quad 0 < a < 1$

2.0 Applications and analysis of linear, quadratic, and cubic equations.

- Homework §1.3 #47-55, 67, 69, §1.4 #111, 112, 116 (see example 8), 125, §2.2 #89, 90, 97, 98, §2.4 #65.
- §1.3 **Discount** The price of a shirt has been discounted %20. The price is $30 after the discount. Find the original price of the shirt.
- §1.3 **Dimensions of a Room** A room is 2 times longer than it is wide, and its perimeter is 24 meters.
  1. Draw a diagram and assign variables for length and width.
  2. Write equations that relate the length and width.
  3. Find the dimensions of the room.

- §1.3 **Course Grade** To get an A in a course, you must have an average of at least 90 on four tests, each out of 100 points. The scores on your first three tests were 88, 94, and 86. What must you earn on the fourth test in order to get an A in the course?
- §1.3 **Travel Time** Two cars start at an interstate interchange and travel in the same direction at an average speeds of 65 mph and 80 mph. How much time must elapse before the two cars are 5 mi apart?
- Go through the rest of the homework problems, but with the numbers changed.

2.6 Combinations of Functions.

- Homework §2.6 #5-23, 31-41
- Sum, Difference, Product, Quotient of Functions
- Do some examples
- Find the Domain of the Quotients of Functions f/g and g/f.

$$f(x) = \sqrt{x}, g(x) = \sqrt{4 - x^2}$$

- Composition of Functions. Do examples.
- Find the domain of composite functions

  Example 1 $f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$. The domain of $f \circ g$ is $[-3, 3]$.

2.7 Inverse Functions.

- Homework §2.7 #1-17, 25-31, 39-47
- The inverse relation of a function f, denoted f^{-1} is found by interchanging the x and y variables. It is a function if f is one-to-one.

- Finding inverse functions informally.

  1. $f(x) = 4x$.
  2. $g(x) = x + 1$

- The functions f and f^{-1} are inverse functions if and only if $f(f^{-1})(x) = x$ and $f^{-1}(f(x)) = x$.

- Verify that the following are inverse functions.

$$f(x) = \frac{5}{x-2}, h(x) = \frac{5}{x} + 2$$

- The graph of f and f^{-1} are reflections of each other about the line $y = x$.

- One-to-one functions. A function is one-to-one if for every value in the range, there is a unique corresponding value in the domain.

- The Horizontal Line Test: A function is one-to-one no horizontal line intercepts the graph at more than one point.

- Finding the inverse function algebraically.

3 Polynomial Functions

3.1 Quadratic Functions and Models

Homework §3.1 #1-27, 37-55, 65-83

The Graph of a Quadratic Function

A quadratic function is a second degree polynomial. Its general form is given by

$$f(x) = ax^2 + bx + c$$

The most basic quadratic function is $y = x^2$. The graph of this function is a parabola.

We can begin with the graph of $y = x^2$, and transform the graph through shifts, reflections, stretches, and compressions.

Example 1 Sketch the graph of the following quadratic functions.

1. $y = (x-2)^2$
2. $y = x^2 + 1$
3. $y = (x-1)^2 - 2$
4. $y = 3x^2$

The **standard form** of a parabola is given by

$$y = a(x-h)^2 + k$$

The graph has the following characteristics.

1. vertex (h, k)
2. facing up if $a > 0$, down if $a < 0$
3. stretched if $|a| > 1$, compressed if $0 < |a| < 1$

Example 2 Sketch the graph of the following quadratic functions. State the vertex.

1. $y = 2(x-3)^2 - 5$
2. $y = -(x+1)^2 + 1$

A quadratic function can be put into standard form by what is known as completing the square. This algebraic technique will be shown in the following example.

Example 3 Write the quadratic function in standard form by completing the square.

1. $y = x^2 + 6x + 1$

   Solution:

$$y = (x^2 + 6x \quad) + 1$$

$$y = (x^2 + 6x + 9 - 9) + 1$$

$$y = (x^2 + 6x + 9) - 9 + 1$$

$$y = (x^2 + 6x + 9) - 8$$

$$y = (x + 3)^2 - 8$$

2. $y = -x^2 + 8x + 7$

The x-intercepts of a quadratic function are found by setting the function equal to zero and solving for x.

Example 4 Find the x-intercept of the function $y = x^2+8x+7$. Next, write the function in standard form and sketch the graph. Label the x-intercepts and the vertex.

Example 5 Find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given x-intercepts.

$$(-5, 0), (5, 0)$$

Solution:

The upward facing parabola can be given by

$$y = (x - (-5))(x - 5)$$

$$y = x^2 - 25$$

The downward facing parabola can be given by

$$y = -(x - (-5))(x - 5) = -(x^2 - 25)$$

$$y = -x^2 + 25$$

3.2 Polynomial Functions of Higher Degree; The Intermediate Value Theorem

Homework §3.2 #1-21, 41-51, 57-61, 67-77.

The Graph of Polynomial Functions

- The graph of a polynomial is **continuous**. This means that it is not necessary to lift the pen when sketching the graph.
- The graph of a polynomial has only smooth, round curves.
- The graph of power functions, $y = x^n$. If n is even, the graph is like a parabola. If n is odd, the graph is like $y = x^3$.
- **Example 1** Sketch the graph of the polynomials.
  1. $f(x) = -x^5$
  2. $h(x) = (x + 1)^4$.
- **The Lead Coefficient Test** It is important to know the behavior of a polynomial as x increases to ∞ or $-\infty$. In general, as x becomes very large. The graph either goes up to infinity, or down to negative infinity. The only question is whether it will become a large negative number or a large positive number.
- **Example 2** : Let $f(x) = x^4$.
  1. As $x \to \infty, f(x) \to \infty$.
  2. As $x \to -\infty, f(x) \to \infty$.
- **Example 3** Let $f(x) = x^5$.
  1. As $x \to \infty, f(x) \to \infty$.
  2. As $x \to -\infty, f(x) \to -\infty$.

- As $x \to \infty$, a polynomial behaves like its highest degree term.

- **Example 4** Let $f(x) = x^5 + 4x^4 - 5x^2 + 9$. Give the behavior of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. Your answer should either be ∞ or $-\infty$.

  1. As $x \to \infty, f(x) \to$ ____.
  2. As $x \to -\infty, f(x) \to$____.

- **Example 5** Let $f(x) = x^6 + 4x^5 - 5x^2 + 9$. Give the behavior of $f(x)$ as $x \to \infty$ and as $x \to -\infty$. Your answer should either be ∞ or $-\infty$.

  1. As $x \to \infty, f(x) \to$ ____.
  2. As $x \to -\infty, f(x) \to$____.

- When sketching the graph, $f(x) \to \infty$ means the graph goes up, while $f(x) \to -\infty$ means the graph goes down.

Zeros of Polynomial Functions

- The Fundamental Theorem of Algebra says that an nth degree polynomial has at most n zeros.

- It can be shown that the graph of a nth degree polynomial has at most $n - 1$ turning points (also called relative maximum and minimum).

- **Real Zeros of Polynomial Functions** If f is a polynomial and a is a real number, the following statements are equivalent.

  1. $x = a$ is a zero of the function.
  2. $x = a$ is a solution of $f(x) = 0$
  3. $(x - a)$ is a factor of the polynomial
  4. $(a, 0)$ is an x-intercept of the graph of f

- **Example 6** Find all of the real zeros of

$$-2x^4 + 2x^2$$

  Solution: $x = 0, x = 1, x = -1$.

- **Repeated Zeros** A factor $(x-a)^k$ yields a repeated zero at $x = a$ of multiplicity k.

  1. If k is odd, the graph crosses the x-axis at $x = a$
  2. If k is even, the graph touches the x-axis at $x = a$

  This is true because close to the number a, the function behaves like the factor $(x-a)^k$, which in turns either behaves like $y = x^2$ if k is even, or like $y = x^3$ if k is odd.

Sketching the Graph of a Polynomial

- **Example 7** Sketch the graph of the polynomial $f(x) = 3x^4 - 4x^3$.

  Solution:

  – Step 1: Determine the behavior of $f(x)$ as $x \to \pm\infty$.

    As $x \to \infty$, $f(x)$ behaves like $3x^4$, and $3x^4 \to \infty$. Therefore $f(x) \to \infty$.

    As $x \to -\infty$, $f(x)$ behaves like $3x^4$, and $3x^4 \to \infty$. Therefore $f(x) \to \infty$.

  – Step 2: Find the zeros of the polynomial and there multiplicity. State whether the graph will touch or cross the x-axis

$$3x^4 - 4x^3 = 0$$
$$x^3(3x - 4) = 0$$

$x = 0$, multiplicity 3, cross. $x = 4/3$, multiplicity 1, cross.

- Step 3: Make a chart showing where f is positive or negative. Plot points.

positive	‖	negative	‖	positive
	$x = 0$		$x = 4/3$	

- Step 4: Sketch the graph.

- **Example 8** Sketch the graph of the polynomial

$$f(x) = 3x^3 - 15x^2 + 18x$$

The Intermediate Value Theorem

Now we will give a theorem which is very important, but at the same time very elementary.

Here is the idea. Suppose that f is a continuous function. Moreover, suppose that $f(1)$ is a negative number and that $f(10)$ is a positive number. Then somewhere between $x = 1$ and $x = 10$, the function f must be equal to zero.

Here is another example. If it is 68° outside at 7 a.m. and 74° outside at 2 p.m., then at some time between 7 a.m. and 2 p.m., the temperature must have been 70°.

If you drive your car on the 5 freeway from Orange County to San Diego, then on your way, you have to pass through Camp Penalton Marine Base.

All of these examples are illustrations of the Intermediate Value Theorem which we now state.

The Intermediate Value Theorem Suppose that f is continuous on a closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Corollary If f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ are both nonzero and have opposite signs (one is positive, the other negative), then there is a number c in the interval (a, b) such that $f(c) = 0$.

4 Rational Functions and Conics

4.1 Rational Functions and Asymptotes

Homework: §4.1 #1-35.

Introduction

A rational function is a fraction of two polynomial functions.

Some examples of rational functions are

1. $y = \dfrac{1}{x}$

2. $y = \dfrac{x^2 + 6x + 5}{x^2 + 5x + 6}$

Domain

The domain of a polynomial is all real numbers except those numbers that will result in a zero denominator.

Example 1 State the domain of the following function.

$$y = \frac{x^2 + 6x + 5}{x^+ 5x + 6}$$

Example 2 :

The first thing to do is, if possible, factor the polynomials in the rational function.

$$y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6} = \frac{(x + 1)(x + 5)}{(x + 2)(x + 3)}$$

The function is undefined when the denominator is equal to 0.

$$(x + 2)(x + 3) = 0$$

Therefore the <u>undefined</u> numbers are $x = -2$ and $x = -3$. Therefore,

$$\begin{aligned} \text{Dom } f &= \{x \in \mathbf{R} | x \neq -2 \text{ and } x \neq -3\} \\ &= \text{All real numbers except } x = 2 \text{ and } x = 3 \end{aligned}$$

Vertical Asymptotes

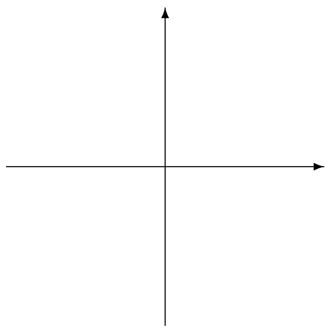
What is the behavior of a rational function when the x-coordinate is near one of these undefined numbers?

Let's study the function $f(x) = \dfrac{1}{x}$. The domain of this function is all real numbers, $x \neq 0$.

Below is a table of values.

x	$f(x) = \frac{1}{x}$
10	1/10
5	1/5
2	1/2
1	1
1/2	2
1/4	4
1/10	10
1/100	100
1/1000	100
−10	−1/10
−5	−1/5
−2	−1/2
−1	−1
−1/2	−2
−1/4	−4
−1/10	−10
−1/100	−100
−1/1000	−100

We can now sketch the graph.



As x approaches the undefined number, 0, from the right, we see that the y-coordinate goes up through the roof. That is,

$$\text{As } x \to 0^+, \quad y \to \infty$$

As x approaches the undefined number, 0, from the left, we see that the y-coordinate goes up through the floor. That is,

$$\text{As } x \to 0^-, \quad y \to -\infty$$

The line $x = 0$ is called a vertical asymptote of the function $y = 1/x$. As the graph approaches the vertical asymptote, the graph will either go through the roof, that is, $f(x) \to \infty$, or the graph goes through the floor, that is, $f(x) \to -\infty$.

The way to find out whether the graph goes through the floor or through the roof is to plot points, but you do not need to plot that many points. You only need to plot enough points to determine whether the function is positive or negative near the vertical asymptote.

Vertical asymptotes always occur at undefined numbers. However, some undefined numbers do not give vertical asymptotes.

Example 3 Find the domain of the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Sketch the graph.

Example 4 :

We find the undefined numbers of the function. The denominator equals zero when

$$x - 1 = 0$$

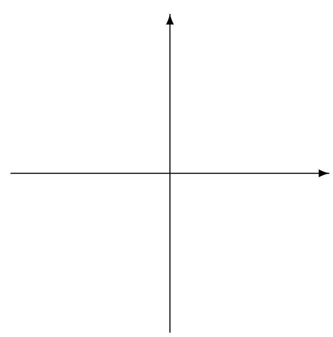
$$x = 1$$

The domain is the set of all real numbers $x \neq 1$.

To sketch the graph, let's first factor the polynomial in the numerator of the function.

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{x-1} = x + 1, \quad x \neq 1$$

The graph of this function is the same as the graph of $y = x + 1$ except that it has a hole at the point $(1, 2)$.



This rational function has an undefined number $x = 1$, but it does not have a vertical asymptote.

Zeros of Rational Functions

To find the zeros of a rational function, set the numerator equal to zero and then solve for x. It is a good idea to find the domain of the function first, because it is possible that a number will make both the numerator and the denominator equal to zero. In this case, the number is not in the domain of the function and is not a zero of the function.

Example 5 Find the zero of the rational function

$$y = \frac{x^2 - 5x - 6}{x^2 - 1}$$

SOLUTION Set the numerator equal to zero and solve for x.

$$\begin{aligned} x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x = 6, x = -1 & \end{aligned}$$

Horizontal Asymptote

Let's return to the graph of $y = 1/x$. We notice that as x becomes a large number, the graph of the function gets very close to the x-axis. It never touches the x-axis because the y-coordinate is never 0. This is because the numerator of the function $y = 1/x$ is never 0.

We can write

$$\text{As } x \to \infty, \quad y \to 0$$

and

$$\text{As } x \to -\infty, \quad y \to 0$$

The function $y = 1/x$ is said to have a **horizontal asymptote** at the line $y = 0$, that is, the x-axis.

To find the horizontal of a rational function, we will use the following principle.

> For large values of x, a polynomial behaves like its highest power term.

Example 6 Find the horizontal asymptote (if one exists) of the function

$$y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6}$$

Solution For finding the horizontal asymptote, factoring the polynomials in the function does not help us. Instead, we will use the principle given above.

$$\text{As } x \to \infty, \quad y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6} \to \frac{x^2}{x^2} = 1$$

The horizontal asymptote is the line $y = 1$.

If a horizontal asymptote exists for a rational function, then we only need to look at the behavior as $x \to \infty$. We will get the same answer as $x \to -\infty$.

Example 7 Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^2 + 6x + 5}{7x^2 + 5x + 6}$$

Solution

$$\text{As } x \to \infty, \quad y = \frac{5x^2 + 6x + 5}{7x^2 + 5x + 6} \to \frac{5x^2}{7x^2} = \frac{5}{7}$$

The horizontal asymptote is the line $y = \frac{5}{7}$.

Example 8 In this example, the degree of the denominator is greater than the degree of the numerator.

Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^2 + 6x + 5}{x^3 + 5x + 6}$$

Solution

$$\text{As } x \to \infty, \quad y = \frac{5x^2 + 6x + 5}{x^3 + 5x + 6} \to \frac{5x^2}{x^3} = \frac{5}{x} \to 0$$

The horizontal asymptote is the line $y = 0$, the x-axis

If the denominator of a rational function has higher degree than the numerator, then the function will always have as a horizontal asymptote the x-axis.

Example 9 In this example, the degree of the numerator is greater than the degree of the denominator.

Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^3 + 6x + 5}{x^2 + 5x + 6}$$

SOLUTION

$$\text{As } x \to \infty, \quad y = \frac{5x^3 + 6x + 5}{x^2 + 5x + 6} \to \frac{5x^3}{x^2} = 5x \to \infty$$

There is no horizontal asymptote. As $x \to \infty$, the graph of the function will go through the roof. This is a case where we need to find the behavior of the graph as $x \to -\infty$.

$$\text{As } x \to -\infty, \quad y = \frac{5x^3 + 6x + 5}{x^2 + 5x + 6} \to \frac{5x^3}{x^2} = 5x \to -\infty$$

If the numerator of a rational function has higher degree than the denominator, then the function will not have a horizontal asymptote. For large values of x, it will behave like a polynomial. As x becomes large, the graph will either go to infinity (through the roof) or it will go to negative infinity (through the floor). As with a polynomial, it is necessary to check the behavior as x goes to both positive infinity and negative infinity.

4.2 Graphs of Rational Functions

Homework §4.2 #1-23, 47-53

The goal of this section is to be able to make a rough sketch of a rational function. Let's go straight to an example.

Example 1 Let $f(x) = \dfrac{x-1}{x+2}$.

1. State the domain of f.

   SOLUTION We set the denominator equal to zero and solve for x.

   $$\begin{aligned} x+2 &= 0 \\ x &= -2 \end{aligned}$$

   The domain of f is all real numbers except $x = -2$.

   The line $x = -2$ will be a vertical asymptote.

2. Find the zeros of f.

   SOLUTION

   Now we set the numerator to 0 and solve for x.

   $$\begin{aligned} x-1 &= 0 \\ x &= 1 \end{aligned}$$

   Therefore, the x-intercept is $(1, 0)$.

3. Find a horizontal asymptote, if it exists.

   SOLUTION

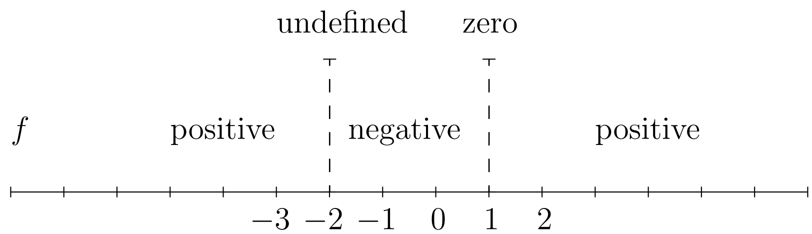
   $$\text{As } x \to \infty, \quad \frac{x-1}{x+2} \to \frac{x}{x} = 1$$

   The function f has horizontal asymptote $y = 1$.

4. Make a chart showing where the function is positive and negative.

   SOLUTION

   The chart is really an number line. We break the number line into intervals whose endpoints are either zeros of the function or undefined numbers. A function will only change signs at either a zero or at an undefined number. We want to determine whether the function is positive or negative on these intervals. To do this, we can plot points.



x	$f(x) = \frac{x-1}{x+2}$	sign
-3	$\frac{-3-1}{-3+2} = \frac{-4}{-1} = 4$	positive
0	$\frac{0-1}{0+2} = \frac{-1}{2}$	negative
2	$\frac{2-1}{2+2} = \frac{1}{4}$	positive

5. Make a rough sketch of the graph.

   SOLUTION

   We have enough information to sketch the graph. We know that there is a vertical asymptote $x = -2$. We can use the chart of positive and negative to determine whether the graph goes up through the roof or down through the floor as x approaches the number -2. We see that f is positive when x is left of -2. Therefore as x approaches -2 from the left, the graph goes to infinity. We see that

f is negative to the right of x. Therefore, the graph goes to $-\infty$ as x approaches x from the right.

5 Exponential and Logarithmic Functions

5.1 Exponential Functions and Their Graphs.

- Homework §5.1 #1-9, 27-31, 45-61.
- Definition: The exponential function f with base a is denoted by

$$y = a^x$$

  where $a > 0$, $a \neq 1$, and x is any real number.
- Evaluate the following exponential functions.
  1. $f(x) = 2^x, \quad x = 5$
  2. $f(x) = 2^{-x}, \quad x = 3$
- The graph of $y = a^x$ for $a > 1$
- The graph of $y = a^{-x}$ for $a > 1$
- Using the one-to-one property to solve an equation involving exponents.
  1. $9 = 3^{x+1}$
  2. $(\frac{1}{2})^x = 8$
- Transformations of graphs of exponential functions
- The natural base e
- Evaluate $y = e^x$ for $x = 1, 2,$ etc.
- Graphing the exponential function with base e.
- Applications: Compounded Interest.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we let $m = n/r$ then we can write this as

$$A = P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}$$

As $n \to \infty$, $(1 + 1/m)^m \to e$.

- Define interest compounded continuously as

$$A = Pe^{rt}$$

- Example: A total of \$12,000 is invested at a rate of 9%. Find the balance after 5 years if it is compounded quarterly, monthly, and continuously.

5.2 Logarithmic Functions and Their Graphs.

- Homework §5.2 #1-21, 27-59, 65-71, 79-85.
- Definition of Logarithmic Function with Base a:

  For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function given by $f(x) = \log_a x$ is called the logarithmic function with base a.

- Evaluate the logarithms

  1. $f(x) = \log_2 x, \quad x = 32$
  2. $f(x) = \log_3 x, \quad x = 1$

- The common logarithmic function. Evaluate on a calculator.
- Properties of Natural Logs

  1. $\log_a 1 = 0$
  2. $\log_a a = 1$
  3. $\log_a a^x = x$ and $a^{\log_a x} = x$
  4. If $\log_a x = \log_a y$ then $x = y$

- Use the properties of logs to simplify: $\log_4 1$, $\log_7 7$, $6^{\log_6 20}$
- Use the one-to-one property to solve for x.

$$\log_3 x = \log_3 12$$

- Graphs of Log Functions. Discuss Domain, range, x-intercept, increasing, one-to-one, vertical asymptote, continuous, reflection of $y = a^x$
- Shifts of Log graph.
- Natural Log Function

  The function $f(x) = \log_e x = \ln x$, $x > 0$ is called the natural logarithmic function.
- Evaluate the Natural Logarithmic Function for $x = .5, 1, 2,$ etc.
- Properties of Natural Logs
  1. $\ln 1 = 0$
  2. $\ln e = 1$
  3. $\ln e^x = x$
  4. $\ln e^x = x$ and $e^{\ln x} = x$
  5. If $\ln x = \ln y$ then $x = y$
- Simplify using the properties of Logs.
  1. $\ln 1/e$
  2. $e^{\ln 5}$
  3. $\frac{\ln e}{3}$
  4. $2 \ln e$
- The domain of the log function.
  1. $y = \ln(x - 2)$
  2. $y = \ln(2 - x)$
  3. $\ln x^2$

5.3 Properties of Logarithms

- Homework §5.3 #1-79
- Change of Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- Rewrite the logarithm as a ratio (a) common logarithms and (b) natural logarithms.

$$log_3 x$$

- Evaluate the logarithm using the change-of-base formula. Round your answer to three decimal places.

$$log_2 0.125$$

- Properties of Logarithms

  1. $\log_a AB = \log_a A + \log_a B$
  2. $\log_a \frac{A}{B} = \log_a A - \log_a B$
  3. $\log_a A^t = t \log_a A$

- Use properties of logarithms to rewrite and simplify the logarithm expression.

$$log_2(4^2 \cdot 3^4)$$

- Find the exact value of the logarithmic expression without using a calculator.

$$log_6 \sqrt[3]{6}$$

- Use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms.

$$\log \frac{4x^2}{y^5 z}$$

- Condense the expression to the logarithm of a single quantity.

$$2 \ln 8 + 5 \ln(z - 4)$$

5.4 Exponential and Logarithmic Equations

- Homework §5.4 #1-53, 75-97

- Determine whether each x-value is a solution (or an approximate solution) of the equation.

  1. $2^{3x+1} = 32$
     (a) $x = -1$
     (b) $x = 2$
  2. $\log_2(x + 3) = 10$
     (a) $x = 1021$
     (b) $x = 17$
     (c) $10^2 - 3$

- Solve for x.

  1. $3^x = 243$
  2. $\ln x = -7$

- Approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically to verify your approximation.

  1. $f(x) = 27^x, \quad g(x) = 9$

2. $f(x) = \ln(x-4)$, $g(x) = 0$

- Solve the exponential equation algebraically. Approximate the result to three decimal places.

1. $2(5^x) = 32$
2. $e^(2x) = 50$
3.
4. $-14 + 3e^x = 11$

- Solve the logarithmic equation algebraically. Approximate the result to three decimal places.

1. $\ln x = 2$
2. $\ln 4x = 1$
3. $\ln x + \ln(x+1) = 1$
4. $\log_4 x - \log_4(x-1) = \frac{1}{2}$

5.5 Exponential and Logarithmic Models

- Homework §5.5 #35-42

- Exponential Growth: Digital Televisions.

  Population growth is often modeled by an exponential function. Here the data for the growth of digital televisions is given.

Year	Households
2003	44.2
2004	49.0
2005	55.5
2006	62.5
2007	70.3

  If we plot these points, one can make an argument that the growth of digital televisions can be modeled by an exponential function.

In general, an exponential grow model is of the form:

$$y = y_0 e^{kt}$$

Here t is time, y_0 is the initial amount, and k is a constant.

In this particular situation, the following function fits the data (how well?).

$$D = D_0 e^{0.1171t}, \quad 3 \leq t \leq 7$$

Here D is the number of households in millions and $t = 3$ represents the year 2003.

We can compare this model with the actual data.

Year	2003	2004	2005	2006	2007
Households	44.2	49.0	55.5	62.5	70.3
Model	43.9	49.4	55.5	62.4	70.2

According to this model, when will the number of U.S. households with digital televisions reach 100 million?

Solution Let $D = 100$. Solve for t. Answer: $t \approx 10.0$. Therefore, the year 2010.

- Modeling Population Growth. In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. Assume that initially there are 100 flies and 2 days there are 300 flies. Find the exponential growth constant k and write a function that models the population growth. How many will there be after 5 days?

  The population model is given by

  $$y = y_0 e^{kt} = 100e^{kt}$$

  At $t = 2$ days, $y = 300$. Therefore

  $$300 = 100e^{2k} \Rightarrow k = \frac{\ln 3}{2}.$$

  We can now write the function

$$y = 100e^{\frac{t \ln 3}{2}}.$$

After 5 days, the population of flies will be

$$y(5) = 100e^{\frac{5 \ln 3}{2}} \approx 1560 \text{ flies}.$$

- Exponential Decay: In living creatures, the ratio of carbon 14 to carbon 12 is about 1 to 1012. When organic material dies, its carbon 12 remain fixed while the carbon 14 will decay with a half-life of 5700 years. The following formula can be used to estimate the age of dead organic material.

$$R = \frac{1}{10^{12}} e^{-t/8223}$$

Estimate the age of a newly discovered fossil in which the ratio of carbon 14 to carbon 12 is

$$R = \frac{1}{10^{13}}$$

Solution $t \approx 18,934$ years.

6 Systems of Equations and Inequalities

6.1 Linear and Nonlinear Systems of Equations

- Homework §6.1 #15-18
- The Substitution Method

1.

$$2x + y = 5$$
$$3x - 2y = 4$$

2.
$$x + y = 4$$
$$x - y = 2$$

6.2 Two-Variable Linear Systems

- Homework §6.2 #1-25
- The Method of Elimination

1.
$$3x - 5 = 7$$
$$-3x - 2y = -1$$

2.
$$3x + 2y = 4$$
$$5x - 2y = 8$$

3.
$$2x - 3y = -7$$
$$3x + y = -5$$

4.
$$5x + 3y = 9$$
$$2x - 4y = 14$$

- Graphical Interpretation of Solutions

Number of Sol	Graphical Interpretation	Slope
Unique Solution	Lines intersect at point	Slopes not equal
Infinitely solutions	Lines coincide	Slopes of are same
No solution	Lines are parallel	Slopes are equal.

- Example: No solution.

$$x - 2y = 3$$
$$-2x + 4y = 1$$

- Infinite solutions

$$2x - y = 1$$
$$4x - 2y = 2$$

7 Matrices and Determinants

7.2 Operations with Matrices

- Homework §7.2 #1-18, 27-34, 41-46, 51-58 (a)
- Definition: An $m \times n$ matrix is a rectangular array with m rows and n columns. The a_{ij} entry is the entry in the ith row and jth column. A matrix of m rows and n columns is said to be of order $m \times n$. If $m = n$ the matrix is square of order n. For square matrices, the entries $a_{11}, a_{22}, \ldots$ are the main diagonal entries.
- Addition of matrices.
- Subtraction of matrices.
- Scalar multiplication of matrices.
- Multiplication of a matrix and a column vector.
- Multiplication of matrices.
- Writing a system of equations as a matrix equation.

7.4 Determinants of a 2×2 matrix

- Homework §7.4 #1-16, 37-52
- The determinant of a 2×2 matrix.

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ab - cd$$

- Do examples.

- Minors and Cofactors Definition: If A is a square matrix, the minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A. The cofactor C_{ij} of the entry a_{ij} is

  $$C_{ij} = (-1)^{i+j} M_{ij}.$$

  Notice that $(-1)^{i+j}$ is either equal to 1 or -1. It is easier to make a sign pattern for the cofactors.

  Example of 3×3 sign pattern:

  $$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

- Example: Find the Minors and Cofactors of a Matrix.

  $$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

  $$\begin{array}{lll} M_{11} = -1 & M_{12} = -5 & M_{13} = 4 \\ M_{21} = 2 & M_{22} = -4 & M_{23} = -8 \\ M_{31} = 5 & M_{32} = -3 & M_{33} = -6 \end{array}$$

  $$\begin{array}{lll} C_{11} = -1 & C_{12} = 5 & C_{13} = 4 \\ C_{21} = -2 & C_{22} = -4 & C_{23} = 8 \\ C_{31} = 5 & C_{32} = 3 & C_{33} = -6 \end{array}$$

- The Determinant of a Square Matrix.

  If A is a square matrix the determinant of A is the sum of the entries in any row or column of A multiplied by their respective cofactors. This method is called expanding by cofactors.

7.5 Applications of Matrices and Determinants: Cramer's Rule

- Homework §7.5 #1-10

- We will solve matrix equations of the form $Ax = b$ using Cramer's Rule.

  We define A_i as the matrix obtained by replacing the ith row of A by the column vector b.

  Cramer's Rule: If A is a square matrix with nonzero determinant $|A|$ then the matrix equation $Ax = b$, where b is a column vector has solution

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|}, \dots\ x_n = \frac{|A_n|}{|A|}$$

- Solve the following systems using Cramer's Rule.

1.
$$4x - 2y = 10$$
$$3x - 5y = 11$$

2.
$$-x + 2y - 3z = 1$$
$$2x + z = 0$$
$$3x - 4y + 4z = 2$$

3.
$$4x - 2y + 3z = -2$$
$$2x + 2y + 5z = 16$$
$$8x - 5y - 2z = 4$$

7.1 Solving Systems of Equations Using Guassian Elimination

Homework §7.1 #51-61

Begin with a system of equations

$$2x + 3y = 5$$

$$3x + y = 1$$

We can solve this using the elimination method. Under the elimination method, we are allowed to multiply an equation by a constant, and we are allowed to add the rows together to derive a new equation.

We can create a matrix equation from the above system.

$$\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

From this, we create what we call an augmented matrix.

$$\begin{bmatrix} 2 & 3| & & 5 \\ 3 & 1|1 & & \end{bmatrix}$$

We use this matrix to do the elimination method. The goal is to write the matrix in row-echelon form.

A matrix in row-echelon form has the following properties.

1. Any rows consisting of zeros occur at the bottom of the matrix.
2. For each row that does not consist of zeros, the first nonzero entry is 1.
3. For two successive nonzero rows, the leading 1 is farther to the left than the leading 1 in the rows below.

A matrix in row-echelon form is in reduced row-echelon form if every column that has a leading 1 has all zeros above and below it.

To put a matrix in row-echelon form, we can use elementary row operations.

1. Interchange any two rows
2. Multiply a row by a constant
3. Add a multiple of one row to another row

We will now use Guassian Elimination with Back-Substitution to solve systems of equations.

1. Write the system as an augmented matrix.
2. Use elementary row operations to put the system in row-echelon form.
3. Write the augmented matrix as a system. Use back substitution to solve.

Example:

$$x - 2y + 3z = 9$$
$$-x + 3y = -4$$
$$2x - 5y + 5z = 17$$

SOLUTION $(1, -1, 2)$
Discuss Inconsistent Equations.

4.3 Conics

- Homework §4.3 #35-41, 61-67, §1.1 #49,51,57-61
- Definition: A conic section is the intersection of a plane and a double-napped cone. The four basic conics are formed when the plane does not pass through the vertex: circle, ellipse, parabola, and hyperbola. If the plane passes through the vertex, a degenerate conic is formed: a point, a line, or two intersecting lines.
- Definition: a parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line, the directrix, and a fixed point, the focus, not on the line. The vertex is the midpoint between the focus and the directrix. The axis of the parabola is the line passing through the focus and the vertex.

- Standard Equation of a Parabola (Vertex at Origin)

  The standard equation of a parabola with vertex $(0,0)$ and directrix $y = -p$ is

  $$x^2 = 4py,\ p \neq 0.$$

  for directrix $x = -p$, the equation is

  $$y^2 = 4px, \quad p \neq 0.$$

  The focus is on the axis p units (directed distance from the vertex.

- Example: Find the focus of a parabola whose equation is $y = -2x^2$.

  SOLUTION

  $$x^2 = 4py$$

  $$y = -2x^2 \Rightarrow x^2 = -\frac{1}{2}y = 4\left(-\frac{1}{8}\right)y.$$

  Therefore $p = -\frac{1}{8}$.

- Ellipses: An ellipse is the set of all points (x, y) in a plane the sum of whose distances from two distinct points (foci) is constant.

- The line through the foci intersect the ellipse at two points (vertices). The chord joining the vertices is the major axis, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the minor axis.

- Standard Equation of an Ellipse centered at the origin with major and minor axes of lengths $2a$ and $2b$, $0 < b < a$ is

  $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

The vertices and foci lie on the major axis, a and c units, respectively, form the center. The numbers a, b, and c satisfy

$$c^2 = a^2 - b^2$$

- Sketch an Ellipse: Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

First, rewrite as (is this necessary?)

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

- Circles: A circle is the set of all points (x, y) in the plane with constant distance r from a point, (h, k), the center.

- Standard Equation of a Circle Centered at the Origin.

$$x^2 + y^2 = r^2$$

If the center is (h, k), the standard equation is

$$(x - h)^2 + (y - k)^2 = r^2$$

- Write the standard form of the equation of a circle with center $(2, -1)$ and radius $r = 3$. Sketch the circle.

- Find the center and radius of the circle. Sketch the graph.

$$(x - 2)^2 + (y + 1)^2 = 25$$

- Hyperbolas: A hyperbola is the set of all points (x, y) in plane the difference of whose distances form two distinct points (foci) is a positive constant.

- The graph has two disconnected parts (branches). The line through the two foci intersect the hyperbola at two points (vertices). The line segment connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola.

- Standard Equation of a Hyperbola (Centered at the Origin):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis horizontal}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis vertical}$$

  The vertices and foci are, respectively, a and c units form the center. Moreover, a, b, and c are related by $b^2 = c^2 - a^2$.

- The asymptotes of a hyperbola: Each hyperbola has two asymptotes that intersect the center. They pass through the vertices of a rectangle with dimensions $2a$ by $2b$.

  The conjugate axis is the line segment joining $(0, b)$ and $(0, -b)$ (or $(-b, 0)$ and $(b, 0)$).

  The Asymptotes are given by

$$y = \pm\frac{b}{a}x \text{ or } y = \pm\frac{a}{b}x.$$

  The key is to draw the box.

- Write the hyperbola in standard form. Sketch the hyperbola, labeling intercepts and drawing asymptotes.

$$4x^2 - y^2 = 16$$

8 Sequences, Series, and Probability

8.1 Sequences and Series

- Homework §8.1 #1-25, 37-49, 59-83
- A sequence is a function is whose domain is the set of positive integers.

  If the domain is of only the first n positive integers, the sequence is said to be finite.
- The terms of a sequence are written as

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

  A sequence is often described as formula.
- Write the first four terms of the sequence given by
  1. $a_n = 3n - 2$
  2. $a_n = 3 + (-1)^n$
- A sequence whose terms alternate in sign. Write the first five terms of the sequence given by

$$a_n = \frac{(-1)^n}{2n-1}$$

- Write an expression for the nth term of the sequence.
  1. $1, 3, 5, 7, \ldots$
  2. $2, -5, 10, -17, \ldots$
- The Fibonacci Sequence: A Recursive Sequence.

  The Fibonacci Sequence is defined as follows:

$$a_0 = 1, a_1 = 1, a_k = a_{k-2} + a_{k-1}, k \geq 2$$

  Write the first six terms in the sequence.

- Factorial Notation. For positive integer n, define n factorial as

$$n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

Evaluate: $0!, 1!, 2!, 3!, 4!$.

- Write the first five terms of the sequence given by

$$a_n = \frac{2^n}{n!}$$

- Evaluate the following factorial expressions.

1. $\frac{8!}{2!6!}$
2. $\frac{2!6!}{3!5!}$
3. $\frac{n!}{(n-1)!}$

- Summation Notation. The sum of the the first n terms of a sequence is represented by

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is called the index of the summation, n is the upper limit, and 1 is the lower limit.

- Find each sum.

1. $\sum_{i=1}^{5} 3i$
2. $\sum_{i=3}^{6} (1 + k^2)$
3. $\sum_{i=0}^{8} \frac{1}{i!}$

- Definition: The sum of the first n terms of a sequence is called a finite series. The sum of an infinite sequence is called an infinite series.

8.5 The Binomial Theorem

- Homework §8.5 #1-9, 15-21
- Binomial Coefficients: A binomial is a polynomial of two terms. We will study the expansion of $(x+y)^n$.
- n choose r is denoted as ${}_nC_r$ or $\binom{n}{r}$.

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Example 1 Evaluate $\binom{5}{3}$.

- The Binomial Theorem: The expansion of $(x+y)^n$ is given by

$$(x+y)^n = x^n + nx^{n-1}y + \ldots \binom{n}{r}x^{n-r}y^r + \ldots nxy^{n-r} + y^n.$$

We can also write

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r$$

This formula is often used as a formula for proofs and abstract work.

Example 2

$$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}x^3y + \binom{4}{4}y^4$$

- Pascal's Triangle is an easier way to expand binomials for relatively small n.